arXiv:1203.5661v1 [math.RT] 26 Mar 2012

A geometric construction of types for the smooth representations of PGL(2) of a local field

Paul Broussous

March 2012

Abstract

We show that almost all (Bushnell and Kutzko) types of $\mathrm{PGL}(2,F)$, F a non-Archimedean locally compact field of odd residue characteristic, naturally appear in the cohomology of finite graphs.

Introduction

Let F be a non-Archimedean locally compact field and G be the group $\mathrm{PGL}(2,F)$. We assume that *the residue characteristic of F is not* 2. In previous works ([2], [3]) we defined a tower of directed graphs $(\tilde{X}_n)_{n\geqslant 0}$ lying G-equivariantly over the Bruhat-Tits tree X of G. We proved the two following facts :

Theorem 1 ([3], Theorem (3.2.4), page 502). *Let $(\pi,\mathcal{V})$ be a non-spherical generic smooth irreducible representation. Then $(\pi,\mathcal{V})$ is a quotient of the cohomology space with compact support $H^1_c(\tilde{X}_{n(\pi)},\mathbb{C})$, where $n(\pi)$ is the conductor of π.*

Theorem 2 ([3], Theorem (5.3.2), page 512). *If $(\pi,\mathcal{V})$ is supercuspidal smooth irreducible representation of G, then we have :*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G\,[H^1_c(\tilde{X}_{n(\pi)},\mathbb{C}),\mathcal{V}] = 1\ .$$

In this paper we make the G-module structure of $H^1_c(\tilde{X}_n,\mathbb{C})$ more explicit for all $n\geqslant 0$, and draw some interesting consequences.

Let us fix an edge $[s_0,s_1]$ of X and denote by $\mathcal{K}_0$ and $\mathcal{K}_1$ the stabilizers in G of s_0 and $[s_0,s_1]$ respectively. Then $\mathcal{K}_0$ and $\mathcal{K}_1$ form a set of representatives of the

two conjugacy classes of maximal compact subgroups in G. If n is even, we have a G-equivariant mapping $p_n : \tilde{X}_n \longrightarrow X$ which respects the graph structures. We denote by Σ_n the subgraph $p_n^{-1}([s_0, s_1])$. If n is odd, then after passing to the first barycentric subdivisions, we have a G-equivariant mapping $p_n : \tilde{X}_n \longrightarrow X$ which respects the graph structures. We denote by Σ_n the subgraph $p_n^{-1}(S(s_0, 1/2))$, where $S(s_0, 1/2)$ denotes the set of points x in X such that $d(x, s_0) \leqslant 1/2$ (here d is the natural distance on the standard geometric realization of X, normalized in such a way that $d(s_0, s_1) = 1$).

Then for all n, Σ_n is a finite graph, equipped with a an action of $\mathcal{K}_1$ if n is even, and $\mathcal{K}_0$ if n is odd. So the cohomology spaces $H^1(\Sigma_n, \mathbb{C})$ provide finite dimensional smooth representations of $\mathcal{K}_1$ or $\mathcal{K}_0$, according to the parity of n.

For each $n \geqslant 0$, we define an finite set $\mathcal{P}_n$ of pairs $(\mathcal{K}, \lambda)$ formed of a maximal compact subgroup $\mathcal{K} \in \{\mathcal{K}_0, \mathcal{K}_1\}$ and of an irreducible smooth representation of $\mathcal{K}$. By definition we have $(\mathcal{K}, \lambda) \in \mathcal{P}_n$ if and only if there exists $k \in \{0, 1, ..., n\}$ such that $(\mathcal{K}, \lambda)$ is an irreducible constituent of the representation $H^1(\Sigma_k, \mathbb{C})$. For $(\mathcal{K}, \lambda) \in \mathcal{P}_n$ and $k \leqslant n$, we denote by m_λ^k the multiplicity of λ in $H_c^1(\Sigma_k, \mathbb{C})$ and we set $m_{n,\lambda} = m_\lambda = n_\lambda^0 + \cdots n_\lambda^n$. Note that n_λ depends on $(\mathcal{K}, \lambda)$ and n.

The main results of this article are the following.

Theorem A. *For all $n \geqslant 0$, we have the direct sum decomposition :*

$$H_c^1(\tilde{X}_n, \mathbb{C}) = \mathbf{St}_G \oplus \bigoplus_{(\mathcal{K},\lambda)\in\mathcal{P}_n} (c\text{-ind}_{\mathcal{K}}^G\ \lambda)^{m_\lambda}\ .$$

(Here $\mathbf{St}_G$ denotes the Steinberg representation of G).

Theorem B. *For all $n \geqslant 0$, any element of $\mathcal{P}_n$ is*

a) *either a type in the sense of Bushnell and Kutzko's type theory [6], which is not a type for the unramified principal series*

b) *or a pair of the form $(\mathcal{K}_0, \chi \circ \det \otimes \mathbf{St}_{\mathcal{K}_0})$, where χ is a smooth character of $F^\times$ of order 2, trivial on the group of 1-units in $F^\times$, and $\mathbf{St}_{\mathcal{K}_0}$ is the representation inflated from the Steinberg representation of* PGL(2) *of the residue field of F,*

c) *or the pair $(\mathcal{K}_1, \mathbf{1}_{\mathcal{K}_1})$, where $\mathbf{1}$ denotes a trivial character.*

Corollary C. *Let $n \geqslant 0$. If $(\mathcal{K}, \lambda) \in \mathcal{P}_n$ is a cuspidal type, then $m_{\lambda,n} = 1$.*

Indeed this follows from Theorems 2 and A using Frobenius reciprocity for compact induction.

By Theorem 1, any Bernstein component of G, different from the unramified principal series component, must have a type in $\mathcal{P}_n$ for n large enough. *Hence*

the graps $\tilde{X}_n$, $n \geqslant 0$, provide a geometric construction of types for almost all Bernstein components of G.

We conjecture that if $(\mathcal{K}, \lambda) \in \mathcal{P}_n$ is a type of G, then $n_\lambda = 1$.

Finally let us observe that this construction gives a new proof that the irreducible supercuspidal representations of G are obtained by compact induction. Our proof differs from original Kutzko's proof ([9], also see [4]) only at the exhaustion steps. Indeed our "supercuspidal" types are the same as Kutzko's, but we prove that any irreducible supercuspidal representation contains such a type by using an argument based on [2] and [3], that is mainly on the existence of the new vector.

The article is organized as follows. The proof of Theorem A relies first on combinatorial properties of the graphs $\tilde{X}_n$ that are stated and proved in §2. Using this combinatorial properties and some homological arguments, we show in §3 how to relate the cohomology of $\tilde{X}_n$ to that of $\tilde{X}_{n-1}$. The irreducible components of $H^1(\Sigma_n)$ are determined in §4 when n is even, and in §5 and §6 when n is odd. A synthesis of the arguments of paragraphs 2 to 6, leading to a proof a theorem A and B, is given in §7.

We shall assume that the reader is familiar with the language of Bushnell and Kutzko's type theory [5] and with the language of strata ([6], [4]).

1 Notation

We shall denote by

F a non-Archimedean non-discrete locally compact field,
$\mathfrak{o}$ its valuation ring,
$\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$,
ϖ the choice of a uniformizer of $\mathfrak{o}$,
$\mathbf{k} = \mathfrak{o}/\mathfrak{p}$ the residue field of F,
p the characteristic of $\mathbf{k}$,
$q = p^f$ the cardinal of $\mathbf{k}$,
G the group $\mathrm{PGL}(2, F)$.
t_ϖ the image of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}$ in G.

The results of this article are obtained under the

Hypothesis. *The characteristic of* $\mathbf{k}$ *is not* 2

We shall often define an element, a subset, or a subgroup of G by giving a (set of) representative(s) in $\mathrm{GL}(2, F)$.

We write T for the diagonal torus of G and $B \supset T$ for the upper standard Borel subgroup. We denote by T^0 the maximal compact subgroup of T, i.e. the set of matrices with coefficients in $\mathfrak{o}^\times$, and by T^n the subgroup of matrices with coefficients in $1+\mathfrak{p}^n$, $n>0$.

Let k, l be integers satisfying $k+l \geqslant 0$. Then $\mathfrak{A}(k,l) = \begin{pmatrix} \mathfrak{o} & \mathfrak{p}^l \\ \mathfrak{p}^k & \mathfrak{o} \end{pmatrix}$ is an $\mathfrak{o}$-order of $\mathrm{M}(2,F)$. We denote by $\Gamma_0(k,l)$ the image in G of its group of units. There are two conjugacy classes of maximal compact subgroups of G. The first one has representative $K = \Gamma_0(0,0)$. A representative $\tilde{I}$ of the second one is the semidirect product of the cyclic group generated by $\Pi = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$ with the Iwahori subgroup $I = \Gamma_0(1,0)$.

The group K is filtered by the normal compact open subgroups

$$K_n = \begin{pmatrix} 1+\mathfrak{p}^n & \mathfrak{p}^n \\ \mathfrak{p}^n & 1+\mathfrak{p}^n \end{pmatrix},\ n \geqslant 1\ .$$

The group I is filtered by the normal compact subgroups I_n, $n \geqslant 1$, defined by:

$$I_{2k+2} = \begin{pmatrix} 1+\mathfrak{p}^{k+1} & \mathfrak{p}^{k+1} \\ \mathfrak{p}^{k+2} & 1+\mathfrak{p}^{k+1} \end{pmatrix},\ I_{2k+1} = \begin{pmatrix} 1+\mathfrak{p}^{k+1} & \mathfrak{p}^{k} \\ \mathfrak{p}^{k+1} & 1+\mathfrak{p}^{k+1} \end{pmatrix},\ k \geqslant 0\ .$$

The subgroups I_n, $n \geqslant 1$, are normalized by Π.

We denote by X the Bruhat-Tits building of G. This is a uniform tree with valency $q+1$. As a G-set and as a simplicial complex X identifies with the following complex. Its vertices are the homothety classes $[L]$ of full $\mathfrak{o}$-lattices L in the vector space $V = F^2$. Two vertices $[L]$ and $[M]$ define an edge is and only if there exists a basis (e_1, e_2) of V such that, up to homothety, we have $L = \mathfrak{o}e_1 \oplus \mathfrak{o}e_2$ and $L = \mathfrak{o}e_1 \oplus \mathfrak{p}e_2$.

The vertices of the standard apartment (i.e. the apartment stabilized by T) are the $s_k = [\mathfrak{o} \oplus \mathfrak{p}^k]$, $k \in \mathbb{Z}$. The element t_ϖ acts as $t_\varpi s_k = s_{k+1}$, $k \in \mathbb{Z}$. The maximal compact subgroup K is the stabilizer of s_0 and $\tilde{I}$ (resp. I) is the global stabilizer (resp. pointwise stabilizer) of the edge $[s_0, s_1]$. If $l \geqslant k$, the pointwise stabilizer of the segment $[s_k, s_l]$ is $\Gamma_0(l, -k)$.

2 Combinatorics of $\tilde{X}_n$

We recall the construction of the directed graphs $\tilde{X}_n$, $n \geqslant 1$.

For any integer $k \geqslant 1$, an *oriented k-path* in X is an injective sequence of vertices $(s_i)_{i=0,\dots,k}$ in X such that, for $k = 0, \dots, k-1$, $\{s_i, s_{i+1}\}$ is an edge in X. We shall allow the index i to run over any interval of integers of length

$k+1$. Let us fix an integer $n \geqslant 1$. The directed graph $\tilde{X}_n$ is constructed as follows. Its edge set (resp. vertex set) is the set of oriented $(n+1)$-paths (resp. n-paths) in X. If $a = \{s_0, s_1, ..., s_{n+1}\}$ is an edge of $\tilde{X}_n$, its head (resp. its tail) is $a^+ = \{s_1, s_2, ..., s_{n+1}\}$ (resp. $a^- = \{s_0, s_1, ..., s_n\}$). The graphs we obtain this way are actually simplicial complex. The group G acts on $\tilde{X}_n$ is an obvious way; the action preserves the structure of directed graph.

When $n = 2m$ is even, we have a natural simplicial projection $p = p_n \ : \tilde{X}_n \to X$ given on vertices by $p(s_{-m}, \dots, s_0, \dots, s_m) = s_0$. It is G-equivariant. Let $e = \{s_0, t_0\}$ be an edge of X. We are going to describe the finite simplicial complex $p^{-1}(e)$. An edge in $\tilde{X}_n$ above the edge e corresponds to an oriented $(2m+1)$-path of one of the following forms:

i) $(s_{-m}, s_{-m+1}, \dots, s_0, t_0, \dots, t_{m-1}, t_m)$

ii) $(t_{-m}, t_{-m+1}, \dots, t_0, s_0, \dots, s_{m-1}, s_m)$

Let $C_{2m-1}(e)$ the set of $(2m-1)$-paths $c = (s_{-m+1}, \dots, s_0, t_0, \dots, t_{m-1})$. We say that $c \in C_{2m-1}(e)$ *lies above* e. Fix $c \in C_{2m-1}(e)$ and consider the simplicial sub-complex $\tilde{X}_{2m}[e, c]$ of $\tilde{X}_{2m}$ whose edges correspond to the $(2m+1)$-paths of the form

$$(a, s_{-m+1}, \dots, s_0, t_0, \dots, t_{m-1}, b)\,.$$

So a (resp. b) can be any neighbour of s_{-m+1} (resp. t_{m-1}) different from s_{-m+2} (resp. t_{m-1}), with the convention that $s_1 = t_0$ and $t_{-1} = s_0$. The simplicial complex $\tilde{X}_{2m}[e, c]$ is connected. It is indeed isomorphic to the complete bipartite graph with sets of vertices:

$$\{a\ ;\ a \text{ neighbour of } s_{-m+1},\ a \neq s_{-m+2}\} \text{ and } \{b\ ;\ b \text{ neighbour of } t_{m-1},\ b \neq t_{m-2}\}\,.$$

Lemma 2.1. *Let e and e' be two edges of X and $c \in C_{2m-1}(e)$, $c' \in C_{2m-1}(e')$. Then $\tilde{X}_{e,c} \cap \tilde{X}_{e',c'} \neq \emptyset$ if and only if we are in one of the following cases:*

i) $e = e'$ and $c = c'$ (so that $\tilde{X}_{2m}[e, c] = \tilde{X}_{2m}[e', c']$);

ii) $e \cap e'$ is reduced to one vertex of X and $c \cup c'$ is an oriented $2m$-path in X. In that case $\tilde{X}_{2m}[e, c] \cap \tilde{X}_{2m}[e', c']$ is reduced to the vertex of $\tilde{X}_{2m}$ corresponding to the $2m$-path $c \cup c'$.

Proof. If $\tilde{X}_{2m}[e, c] \cap \tilde{X}_{2m}[e', c'] \neq \emptyset$, then $e \cap e' = p(\tilde{X}_{2m}[e, c]) \cap p(\tilde{X}_{2m}[e', c']) \neq \emptyset$. Assume first that $e = e'$. Then $c = c'$, for if $c \neq c'$, then $\tilde{X}_{2m}[e, c] \cap \tilde{X}_{2m}[e', c'] = \emptyset$; indeed if $\tilde{s}$ is a vertex of $\tilde{X}_{2m}[e, c]$ then it determines c uniquely. Now assume that $e \cap e'$ is a vertex. Let $\tilde{s} \in \tilde{X}_{2m}[e, c] \cap \tilde{X}_{2m}[e', c']$. Then $\tilde{s}$ contains c and c' as subsequences, with $c \neq c'$. So by a length argument $s = c \cup c'$. Conversely if $c \cup c'$ is an oriented $2n$-path then $c \cup c'$ is a vertex of $\tilde{X}$ lying in $\tilde{X}_{2m}[e, c] \cap \tilde{X}_{2m}[e', c']$.

Corollary 2.2. *For any edge e of X, the connected components of $p^{-1}(e)$ are the $\tilde{X}_{2m}[e, c]$, where c runs over $C_{2m-1}(e)$.*

Define a 1-dimensional simplicial complex Y_{2m-1} in the following way. Its vertices are the connected components $\tilde{X}_{2m}[e, c]$, where e runs over the edges of

X and c over $C_{2m-1}(e)$, and two vertices $\tilde{X}_{2m}[e,c]$ and $\tilde{X}_{2m}[e',c']$ are linked by an edge if they intersect. Note that Y_{2m-1} is naturally a G-simplicial complex.

Corollary 2.3. *As a G-simplicial complex Y_{2m-1} is canonically isomorphic to the complex $\tilde{X}_{2m-1}$.*

Assume that $m \geqslant 1$. We say that an edge of $\tilde{X}_{2m-1}$ *lies above* a vertex s_0 of X if as an oriented $2m$-path it has the form $(s_{-m}, \ldots, s_o, \ldots, s_m)$. For any vertex s_0 of X we write $\tilde{X}_{2m-1}[s_o]$ for the subsimplicial complex of $\tilde{Y}$ formed of the edges lying above s_0.

Lemma 2.5 *When $m = 1$ the simplicial complexes $\tilde{X}_{2m-1}[s_0] = \tilde{X}_1[s_0]$ are connected.*

Proof. We may identify the neighbour vertices of s_0 in X with the points of the projective line $\mathbb{P}^1(\bar{M}) \simeq \mathbb{P}^1(\mathbf{k})$, where $s_0 = [M]$ and $\bar{M} = M/\mathfrak{p}_K M$. The vertices of $\tilde{X}_1[s_0]$ are the oriented 1-paths (s_0, x), (y, s_0), x, $y \in \mathbb{P}^1(\bar{M})$. Two oriented 1-paths of the form (x, s_0) and (s_0, y) are linked by the edge (x, s_0, y). Let (x, s_0), (y, s_0) be two oriented 1-paths with $x \neq y$. Since $|\mathbb{P}^1(\mathbf{k})| \geqslant 3$, there exists $z \in \mathbb{P}^1(\bar{M})$ distinct from x and y. Then (x, s_0) is linked to (s_o, z) via the path (x, s_0, z) and (s_0, z) is linked to (y, s_0) via the path (y, s_0, z). For vertices of the form (s_0, x), (s_0, y) the proof is similar.

We now assume that $m > 1$. We write $C_{2m-2}(s_0)$ for the set $(2m-2)$-paths of the form $(s_{-m+1}, \ldots, s_0, \ldots, s_{m-1})$. For any $c \in C_{2m-2}(s_0)$, we consider the subsimplicial complex $\tilde{X}_{2n-1}[s_0, c]$ of $\tilde{X}_{2m-1}$ whose edges corresponds to the $2m$-paths of the form $(a, s_{-n+1}, \ldots, s_0, s_{n-1}, b)$. We have results similar to lemma 1.2, corollaries 2.2 and 2.3.

Lemma 2.6. i) *For any vertex s_0 of X and for $c \in C_{2m-2}(s_0)$, $\tilde{X}_{2m-1}[s_0, c]$ is connected. It is indeed isomorphic to a complete bipartite graph constructed on two sets of $q = |\mathbf{k}|$ elements.*
ii) *Let s and s' be vertices of X, $c \in C_{2m-2}(s)$ and $c' \in C_{2m-2}(s')$. Then $\tilde{X}_{2m-1}[s, c] \cap \tilde{X}_{2m-1}[s', c'] \neq \emptyset$ if and only if $s = s'$ and $c = c'$, or $\{s, s'\}$ is an edge in X and $c \cup c'$ is an oriented $2n-1$-path. In this last case $\tilde{X}_{2m-1}[s, c] \cap \tilde{X}_{2m-1}[s', c'] = \{\tilde{s}\}$, where the vertex $\tilde{s}$ of $\tilde{X}_{2m-1}$ corresponds to the $(2n-1)$-path $c \cup c'$.*

iii) For any vertex s of X, the connected components of $\tilde{X}_{2m-1}[s]$ are the $\tilde{X}_{2m-1}[s, c]$, c running over $C_{2m-2}(s)$.

We can consider the 1-dimensional simplicial complex Z_{2m-2} whose vertices are the connected components $\tilde{X}_{2m-1}[s, c]$, s running over the vertices of X and c over $C_{2m-2}(s)$, and where two connected components define an edge if and only if they intersect. Note that Z_m is naturally a G-simplicial complex.

Corollary 2.7. *As a G-simplicial complex Z_{2m-2} is isomorphic to X_{2n-2}.*

3 The cohomology of $\tilde{X}_n$: first reductions

If Σ is a locally finite 1-dimensional simplicial complex, we write Σ^0 (resp. $\Sigma^{(1)}$,Σ^1) for its set of vertices (resp. non-oriented edges, oriented edges). We let $C_0(\Sigma)$ (resp. $C_1(\Sigma)$) denote the $\mathbb{C}$-vector space with basis Σ^0 (resp. Σ^1). We define the space $C_c^0(\Sigma,\mathbb{C}) = C_c^0(\Sigma)$ (resp. $C_c^1(\Sigma,\mathbb{C}) = C_c^1(\Sigma)$) of oriented simplicial 0-cochains (resp. 1-cochains) with compact support by:

$C_c^0(\Sigma)$ = space of all linear forms $f : C_0(Z) \to \mathbb{C}$ such that $f(s) = 0$ except for a finite number of vertices s;

$C_c^1(\Sigma)$ = space of all linear forms $\omega : C_1(\Sigma) \to \mathbb{C}$ such that $f([a,b]) = 0$ except for finite number of oriented edges $[a,b]$ and $f([a,b]) = -f([b,a])$.

We set $C_c^k(\Sigma) = 0$ for $k \in \mathbb{Z}\backslash\{0,1\}$ and define a coboundary map $d : C_c^0(\Sigma) \to C_c^1(\Sigma)$ by $df([a,b]) = f(b) - f(a)$. The cohomology of the cochain complex $(C_c^\bullet(\Sigma), d)$ computes the cohomology with compact support $H_c^\bullet(\Sigma,\mathbb{C}) = H_c^\bullet(\Sigma)$ of (the standard geometric realization of) Σ. If Σ is acted upon by a group H whose action is simplicial then $(C_c^\bullet(\Sigma), d)$ is in a straightforward way a complex of H-modules and its cohomology computes $H_c^1(\Sigma)$ as a H-module. When T is finite we drop the subscripts c.

Since the stablizer of a finite number of vertices of X is open in G, we see that for $n \geqslant 1$, the G-modules $C_c^0(\tilde{X}_n)$, $C_c^1(\tilde{X}_n)$ and therefore $H_c^1(\tilde{X}_n)$ are smooth.

In the sequel we fix $m \geqslant 1$ and we abbreviate $\tilde{X}_{2m} = \tilde{X}$. The disjoint union $\tilde{X}^1 = \bigsqcup_{e\in X^{(1)}} \tilde{X}_e$, where $\tilde{X}_e = p^{-1}(e)$, induces an isomorphism:

$$\begin{array}{rccc} (3.1) \qquad & C_c^1(\tilde{X}) & \simeq & \bigoplus_{e\in X^{(1)}} C^1(\tilde{X}_e) \\ & \omega & \mapsto & (\omega_{|C_1(\tilde{X}_e)})_{e\in X^{(1)}} \end{array}$$

Similarly the non-disjoint union $\tilde{X}^0 = \bigcup_{e\in X^{(1)}} \tilde{X}_e^0$ induces an injection:

$$\begin{array}{rrccc} (3.2) \qquad & j : & C_c^0(\tilde{X}) & \hookrightarrow & \bigoplus_{e\in X^{(1)}} C^0(\tilde{X}_e) \\ & & f & \mapsto & (f_{|C_0(\tilde{X}_e)})_{e\in X^{(1)}} \end{array}$$

We have the following commutative diagram of G-modules:

$$\begin{tikzcd}
 & H^0_c(\tilde{X}) \arrow[r] \arrow[d] & \bigoplus_{e\in X^{(1)}} H^0(\tilde{X}_e) \arrow[r, "\varphi"] \arrow[d] & \mathrm{cokerj} \arrow[d, equal] & \\
0 \arrow[r] & C^0_c(\tilde{X}) \arrow[r, "j"] \arrow[d, "d"'] & \bigoplus_{e\in X^{(1)}} C^0(\tilde{X}_e) \arrow[r] \arrow[d, "\oplus d_e"'] & \mathrm{cokerj} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^1_c(\tilde{X}) \arrow[r] \arrow[d] & \bigoplus_{e\in X^{(1)}} C^1(\tilde{X}_e) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & H^1_c(\tilde{X}) \arrow[r] & \bigoplus_{e\in X^{(1)}} H^1(\tilde{X}_e) \arrow[r] & 0 &
\end{tikzcd}$$

Here, for $e \in X^{(1)}$, d_e denote the coboundary map $C^0(\tilde{X}_e) \to C^1(\tilde{X}_e)$. Since $\tilde{X}$ is connected ([2] Lemma 4.1) and non compact, we have $H^0_c(\tilde{X}) = 0$. So *the snake lemma* gives the kernel-cokernel exact sequence:

$$0 \to \bigoplus_{e\in X^{(1)}} H^0(\tilde{X}_e) \to \mathrm{coker}j \to H^1_c(\tilde{X}) \to \bigoplus_{e\in X^{(1)}} H^1(\tilde{X}_e) \to 0$$

that is

$$0 \to \mathrm{cokerj}/\varphi\big(\bigoplus_{e\in X^{(1)}} H^0(\tilde{X}_e)\big) \to H^1_c(\tilde{X}) \to \bigoplus_{e\in X^{(1)}} H^1(\tilde{X}_e) \to 0 \tag{3.3}$$

Abbreviate $Y = Y_{2m-1}$.

Lemma 3.4. *We have a canonical isomorphism of G-modules*

$$\mathrm{cokerj}/\varphi\big(\bigoplus_{e\in X^{(1)}} H^0(\tilde{X}_e)\big) \simeq H^1_c(Y)\,.$$

Proof. From corollary 2.2 we have

$$\bigoplus_{e\in X^{(1)}} C^0(\tilde{X}_e) = \bigoplus_{e\in X^{(1)}} \bigoplus_{c\in C(e)} C^0(\tilde{X}_{e,c})\,.$$

So the map j is given by $f \mapsto \bigoplus_{e\in X^{(1)}} \bigoplus_{c\in C(e)} f_{e,c}$, where $f_{e,c} = f_{|C_0(\tilde{X}_{e,c})}$. Consider the G-equivariant morphism of vector spaces

$$\psi \;:\; \bigoplus_{e\in X^{(1)}} \bigoplus_{c\in C(e)} C^0(\tilde{X}_{e,c}) \to C^1_c(Y)$$

given as follows. If σ is an oriented edge of Y then their exist uniquely determined edges e_o, e'_o of X, $c_o \in C(e_o)$, $c'_o \in C(e'_o)$, such that σ corresponds to the intersection $\tilde{X}_{e_o,c_o} \cap \tilde{X}_{e'_o,c'_o} = \{s_o\}$, $s_o \in \tilde{X}^0$. We then set

$$\psi[(f_{e,c})_{e,c}](\sigma) = f_{e'_o,c'_o}(s_o) - f_{e_o,c_o}(s_o)\,.$$

Then ψ is surjective and its kernel is precisely $j(C_c^0(\tilde{X}))$. So we may identify coker$j$ with $C_c^1(Y)$. From corollary 2.2, we have

$$\bigoplus_{e\in X^{(1)}} H^0(\tilde{X}_e) = \bigoplus_{e\in X^{(1)}} \bigoplus_{c\in C(e)} H^0(\tilde{X}_{e,c})$$

so that we may identify $\bigoplus_{e\in X^{(1)}} H^0(\tilde{X}_e)$ with $C_c^0(\tilde{Y})$. Under our identifications the map $\varphi : \bigoplus_{e\in X^{(1)}} H^0(\tilde{X}_e) \to \mathrm{coker}j$ corresponds to the coboundary map $d : C_c^0(\tilde{Y}) \to C_c^1(\tilde{Y})$, and we are done since all our identifications are G-equivariant.

Proposition 3.5. *For $m \geqslant 1$, we have an isomorphism of G-modules:*

$$H_c^1(\tilde{X}_n) \simeq H_c^1(\tilde{X}_{2m-1}) \oplus c\text{-ind}_{\mathfrak{K}_{e_o}}^G H^1(\tilde{X}_{e_o})$$

for any edge e_o of x and where $\mathfrak{K}_{e_o}$ denotes the stabilizer of e_o in G.

Proof. From the short exact sequence (3.3) and lemma 3.4, we have the exact sequence of G-modules:

$$0 \to H_c^1(Y) \to H_c^1(\tilde{X}) \to \oplus_{e\in X^{(1)}} H^1(\tilde{X}_e) \to 0 \tag{3.6}$$

Since G acts transitively on the edges of X, $\oplus_{e\in X^{(1)}} H^1(\tilde{X}_e)$ identifies with the compactly induced representation $c\text{-ind}_{\mathfrak{K}_{e_o}}^G H^1(\tilde{X}_{e_o})$. Moreover by [Vign ??](**Trouver la bonne référence**) this induced representation is projective in the category of smooth complex representations of G. So the sequence (3.7) splits.

We assume that $m \geqslant 1$ and we abbreviate $\tilde{X} = \tilde{X}_{2m-1}$. The disjoint union $\tilde{X}^1 = \bigsqcup_{s\in X^0} \tilde{X}_s^1$ induces an isomorphism:

$$\begin{array}{ccc} C_c^1(\tilde{X}) & \simeq & \bigoplus_{s\in X^0} C^1(\tilde{X}_s) \\ \omega & \mapsto & (\omega_{|C_1(\tilde{X}_s)})_{s\in X^0} \end{array} \tag{3.7}$$

Similarly the non-disjoint union $\tilde{X}^0 = \bigcup_{s\in X^0} \tilde{X}_s^0$ induces an injection:

$$\begin{array}{cccc} j: & C_c^0(\tilde{X}) & \hookrightarrow & \bigoplus_{s\in X^0} C^0(\tilde{X}_s) \\ & f & \mapsto & (f_{|C_0(\tilde{X}_s)})_{s\in X^0} \end{array} \tag{3.8}$$

We have the following commutative diagram of G-modules:

$$\begin{tikzcd}
& H^0_c(\tilde{X}) \arrow[r] \arrow[d] & \bigoplus_{s\in X^0} H^0(\tilde{X}_s) \arrow[r, "\varphi"] \arrow[d] & \mathrm{cokerj} \arrow[d, equal] & \\
0 \arrow[r] & C^0_c(\tilde{X}) \arrow[r, "j"] \arrow[d, "d"'] & \bigoplus_{s\in X^0} C^0(\tilde{X}_s) \arrow[r] \arrow[d, "\oplus d_s"'] & \mathrm{cokerj} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^1_c(\tilde{X}) \arrow[r] \arrow[d] & \bigoplus_{s\in X^0} C^1(\tilde{X}_s) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
& H^1_c(\tilde{X}) \arrow[r] & \bigoplus_{s\in X^0} H^1(\tilde{X}_s) \arrow[r] & 0 &
\end{tikzcd}$$

Here, for $s \in X^0$, d_s denote the coboundary map $C^0(\tilde{X}_s) \to C^1(\tilde{X}_s)$. By Lemma 2.4, $\tilde{X}$ is connected. So we have $H^0_c(\tilde{X}) = 0$ since $\tilde{X}$ is non-compact. The *snake lemma* gives the kernel-cokernel exact sequence:

$$0 \to \mathrm{cokerj}/\varphi\Big(\bigoplus_{s\in X^0} H^0(\tilde{X}_s)\Big) \to H^1_c(\tilde{X}) \to \bigoplus_{s\in X^0} H^1(\tilde{X}_s) \to 0 \tag{3.9}$$

Lemma 3.10. *We have a canonical isomorphism of G-modules*

$$\mathrm{cokerj}/\varphi\Big(\bigoplus_{s\in X^0} H^0(\tilde{X}_s)\Big) \simeq H^1_c(\tilde{X}_{2m-2})\,.$$

Proof. It is similar to the proof of lemma 3.4 and relies on lemma 2.6 and corollary 2.7.

Proposition 3.11 *For $m \geqslant 1$, we have an isomorphism of G-modules :*

$$H^1_c(\tilde{X}_{2m-1}) \simeq H^1_c(\tilde{X}_{2m-2}) \oplus c\text{-}\mathrm{ind}_{\mathfrak{K}_{s_o}}^G H^1(\tilde{X}_{s_o})$$

for any vertex s_o and where $\mathfrak{K}_{s_o}$ denotes the stabilizer of s_o in G.

Proof. Similar to the proof of proposition 3.5.

Recall [3] that $\tilde{X}_0$ is different from X. This is a directed graph whose set of vertices is isomorphic to X^0 as a G-set and whose set of edges is isomorphic to the G-set of oriented edges of X.

4 Determination of the inducing representations – I

Let $m \geqslant 0$ be a fixed integer and $e_0 = [s_0, s_1]$ be the standard edge. The aim of this section is to determine the $\mathcal{K}_{e_0}$-module $H^1(\tilde{X}_{2m}[e_0])$. Here we have $\mathcal{K}_{e_0} = \tilde{I}$,

the normalizer in G of the standard Iwahori subgroup. We have the semidirect products:

$$\tilde{I} = \langle \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} \rangle \ltimes I = E^\times I$$

for any totally ramified subfield extension $E/F \subset \mathrm{M}(2,F)$ such that $E^\times$ normalizes I.

We first assume that $m \geqslant 1$. By Corollary (2.2), we have the disjoint union:

$$\tilde{X}_{2m}[e_0] = \coprod_{c \in C_{2m-1}(e_0)} \tilde{X}_{2m}[e_0, c]\ .$$

The group $\tilde{I}$ acts transitively on $C_{2m-1}(e_0)$. This comes form the standard fact that I, the pointwise stabilizer of e_0 acts transitively on the apartments of X containing e_0.

Let $c_0 \in C_{2m-1}(e_0)$ be the path

$$s_{-m+1}, ..., s_0, s_1, ..., s_m\ .$$

The global stabilizer of $\tilde{X}_{2m}[e_0, c_0]$ in $\tilde{I}$ is the pointwise stabilizer of c_0 in $\tilde{I}$, that is

$$\Gamma_0(m, m-1) = \begin{pmatrix} \mathfrak{o}^\times & \mathfrak{p}^{m-1} \\ \mathfrak{p}^m & \mathfrak{o}^\times \end{pmatrix} = T^0 I_{2m-1}\ .$$

It follows that

$$H^1(\tilde{X}_{2m}[e_0]) = \mathrm{ind}_{T^0 I_{2m-1}}^{\tilde{I}} H^1(\tilde{X}_{2m}[e_0, c_0])\ . \tag{4.1}$$

On the other hand, an easy calculation shows that the pointwise stabilizer of $\tilde{X}_{2m}[e_0, c_0]$ is $T^1 I_{2m}$, where T^1 is the congruence subgroup of T given by

$$T^1 = \begin{pmatrix} 1+\mathfrak{p} & 0 \\ 0 & 1+\mathfrak{p} \end{pmatrix}\ .$$

So the $T^0 I_{2m-1}$-module $H^1(\tilde{X}_{2m}[e_0, c_0]$ may be viewed as a representation of the finite group $T^0 I_{2m-1}/T^1 I_{2m}$, that is a semidirect product of the cyclic group $\mathbf{k}^\times$ with the abelian group $I_{2m-1}/I_{2m} \simeq \mathbf{k} \oplus \mathbf{k}$.

Set $\Gamma = \tilde{X}_{2m}[e_0, c_0]$. This is a finite directed graph. Let Σ_{-m} (resp. Σ_{m+1}) denote the set of verticed of X that are neighbours of s_{-m+1} and different from s_{-m+2} resp. neighbours of s_m and different from s_{m-1}. Then the vertex set of Γ is

$$\begin{aligned} \Gamma^0 &= \{(a, s_{-m+1}, ..., s_0, ..., s_m)\ ;\ a \in \Sigma_{-m}\} \coprod \{(s_{-m+1}, ..., s_0, ..., s_m, b)\ ;\ b \in \Sigma_{m+1}\} \\ &\simeq \Sigma_{-m} \coprod \Sigma_{m+1} \end{aligned}$$

and its edge set is

$$\Gamma^1 = \{(a, s_{-m+1}, ..., s_0, ..., s_m, b) \ ; \ a \in \Sigma_{-m}, \ b \in \Sigma_{m+1}\} \simeq \Sigma_{-m} \times \Sigma_{m+1} \ .$$

In particular Γ is a bipartite graph based on two sets of q elements. In particular, its Euler character is given by

$$\chi(\Gamma) = 1 - \dim_{\mathbb{C}} H^1(\Gamma) = 2q - q^2 \ ,$$

so that

$$\dim_{\mathbb{C}} H^1(\Gamma) = q^2 - 2q + 1 = (q-1)^2 \ . \tag{4.2}$$

Let $\mathbb{C}[\Gamma^1]$ be the space of complex fonction on Γ^1 and $\mathcal{H}(\Gamma)$ be the space of harmonic 1-cochains on Γ:

$$\mathcal{H}(\Gamma) = \{f \in \mathbb{C}[\Gamma] \ ; \ \sum_{a \in \Gamma^1, \ s \in a} [a : s] f(a) = 0 \ , \ \text{all } s \in \Gamma^0\} \ .$$

Here $[a : s]$ denote an incidence number. In our case :

$$\text{(Harm)} \qquad f \in \mathcal{H}(\Gamma) \text{ iff } \left\{ \begin{array}{rcl} \displaystyle\sum_{a \in \Sigma_{-m}} f(a, s_{-m+1}, ..., s_m, b) & = & 0, \text{ all } b \\ \displaystyle\sum_{b \in \Sigma_{m+1}} f(a, s_{-m+1}, ..., s_m, b) & = & 0, \text{ all } a \end{array} \right.$$

This is a standard result (see e.g. [3]Lemma (1.3.2)), that, as a $T^0 I_{2m-1}/T^1 I_{2m}$-module, $H^1(\Gamma)$ is isomorphic to the contragredient module of $\mathcal{H}(\Gamma)$.

An easy computation shows that we may identify Γ^1 with $\mathbf{k} \times \mathbf{k}$ in such a way that:

1) an element of $I_{2m-1} = \begin{pmatrix} 1 + \mathfrak{p}^m & \mathfrak{p}^{m-1} \\ \mathfrak{p}^m & 1 + \mathfrak{p}^m \end{pmatrix}$ acts as

$$(1 + \begin{pmatrix} \varpi^m a & \varpi^{m-1} b \\ \varpi^m c & \varpi^m d \end{pmatrix}).(x, y) = (x + \bar{b}, y + \bar{c})$$

for $a, b, c, d \in \mathfrak{o}$, $x, y \in \mathbf{k}$, and

2) an element of T^0 acts as

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.(x, y) = (\bar{a}\bar{d}^{-1} x, \bar{d}\bar{a}^{-1} y)$$

and

the condition (Harm) writes:

$$f \in \mathcal{H}(\Gamma) \text{ iff } \left\{ \begin{array}{rcl} \displaystyle\sum_{x \in \mathbf{k}} f(x, y) & = & 0, \ \text{all } y \in \mathbf{k} \\ \displaystyle\sum_{y \in \mathbf{k}} f(x, y) & = & 0, \ \text{all } x \in \mathbf{k} \end{array} \right.$$

A basis of $\mathbb{C}[\Gamma]$ is formed of the fonctions $\chi_1 \otimes \chi_2(x,y) = \chi_1(x)\chi_1(y)$, where, for $i = 1,2$, χ_i runs over the characters of $(\mathbf{k},+)$. It is clear that the $(q-1)^2$ dimensional subspace of $\mathbb{C}[\Gamma]$ generated by the $\chi_1 \otimes \chi_2$, $\chi_1 \not\equiv 1$, $\chi_2 \not\equiv 1$, is contained in $\mathcal{H}(\Gamma)$. So using (4.2), we obtain:

$$\mathcal{H}(\Gamma) = \mathrm{Span}\{\chi_1 \otimes \chi_2 \; ; \chi_i \in \widehat{\mathbf{k}^\times}, \chi_i \not\equiv 1 \; , i = 1,2\} \; . \tag{4.3}$$

It follows from (4.3) that as an I_{2m-1}/I_{2m}-module, the space $\mathcal{H}(\Gamma)$ is the direct sum of 1-dimensional representations corresponding to the characters $\alpha = \alpha(\chi_1, \chi_2)$, $\chi_i \not\equiv 1$, $i = 1,2$, given by

$$\alpha\big(1 + \begin{pmatrix} \varpi^m a & \varpi^{m-1} b \\ \varpi^m c & \varpi^m d \end{pmatrix}\big) = \chi_1(b)\chi_2(a) \; .$$

In particular $\mathcal{H}(\Gamma)$ is isomorphic to its contragredient and therefore isomorphic to $H^1(\Gamma)$ as an I_{2m-1}/I_{2m}-module. In the language of strata (the reader may refer to [4]§4), for $\chi_i \not\equiv 1$, $i = 1,2$, the character $\alpha(\chi_1,\chi_2)$ corresponds to a stratum of the form $[\mathfrak{I}, 2m, 2m-1, \beta]$, where $\mathfrak{I}$ is the standard *Iwahori order* and $\beta \in \mathrm{M}(2,F)$ is an element of the form $\Pi^{2m-1}\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$, $u, v \in \mathfrak{o}^\times$. In the terminology of [4]§4, page 98, this stratum is a *ramified simple stratum.*

We now have enough material to prove the following result.

Proposition (4.4). *Let λ be an irreducible constituent of*

$$H^1(\tilde{X}_{2m}[e_0]) = \mathrm{ind}_{T^0 I_{2m-1}}^{\tilde{I}} H^1(\tilde{X}_{2m}[e_0, c_0]) \; .$$

Then the compactly induced representation $\mathrm{c}-\mathrm{Ind}_{\tilde{I}}\lambda$ *is irreducible, whence supercuspidal.*

Proof. It is a standard result that an irreducible compactly induced representation is supercuspidal (see [10] or [8], page 194).

The proof of the irreducibility is also standard by an argument due to Kutzko. But we repeat it for convenience. By Frobenius reciprocity, the restriction of λ to I_{2m-1} contains a character $\alpha(\chi_1, \chi_2)$ corresponding to a (ramified) simple stratum. Since λ is irreducible and since $\tilde{I}$ normalizes I_{2m-1}, the restriction $\lambda_{|I_{2m-1}}$ is a direct sum $\alpha_1 \oplus \cdots \alpha_r$ of $\tilde{I}$-conjugates of $\alpha(\chi_1,\chi_2)$. They all correspond to simple strata. Let $g \in G$ be an element intertwining λ with itself. Then by restriction it intertwines a character α_i with a character α_j for some $j = 1, ..., r$. By [4]§ Lemma (16.1), page 111, such an element G must belongs to $\tilde{I}$. It follows that the G-intertwining of λ is equal to $\tilde{I}$ and that the representation $\mathrm{c}-\mathrm{Ind}_{\tilde{I}}^G\lambda$ is irreducible according to Mackey's irreducibility criterion ([8] Proposition (1.5), page 195).

We finally consider the case $m = 0$. The directed graph $\tilde{X}_0$ has X^0 as vertex set. An edge $\{t, s\}$ in X gives rise to two edges $[s,t]$ and $[t,s]$ in $\tilde{X}_0$. Since the

action of G on $\tilde{X}_0$ preserves the structure of digraph, the G-module $H_c^1(\tilde{X}_0)$ may be computed using the following complex :

$$0 \longrightarrow C_c^0(\tilde{X}_0) \longrightarrow C_c^{(1)}(\tilde{X}_0)$$

where $C_c^{(1)}(\tilde{X}_0)$ is the space of (unoriented) 1-cochains, that is the space of maps from $\tilde{X}_0^{(1)}$ (unoriented edges) to $\mathbb{C}$ with finite support. The coboundary map is here given by $df[s,t] = f(t) - f(s)$. Consider the G-equivariant injection j : $C_c^1(X) \longrightarrow C_c^{(1)}(\tilde{X}_0)$ given by $j(\omega)$: $[s,t] \mapsto \omega([s,t])$. We have the commutative diagram of G-modules :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_c^0(X) & \xrightarrow{\text{id}} & C_c^0(\tilde{X}_0) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_c^1(X) & \xrightarrow{j} & C^1(\tilde{X}_0) & \longrightarrow & C_c^{(1)}(\tilde{X}_0)/\mathrm{Im} j & \longrightarrow & 0
\end{array}$$

The quotient $C_c^{(1)}(\tilde{X}_0)/\mathrm{Im} j$ identifies with the subspace of $C_c^{(1)}(\tilde{X}_0)$ formed of those functions f satisfying $f([s,t]) = f([t,s])$ for all edges $\{s,t\}$ of X. This subspace is nothing other than the compactly induced representation $\mathrm{c}-\mathrm{Ind}_{\tilde{I}}^G \mathbf{1}_{\tilde{I}}$. The cokernel exact sequence writes:

$$0 \longrightarrow H_c^1(X) \longrightarrow H_c^1(\tilde{X}_0) \longrightarrow c\text{-ind}_{\tilde{I}}^G \mathbf{1}_{\tilde{I}} \longrightarrow 0$$

Now we use the following two facts :

– the representation $\mathrm{c}-\mathrm{Ind}_{\tilde{I}}^G \mathbf{1}_{\tilde{I}}$ is a projective object of the category of smooth representations of G,

– the G-module $H_c^1(X)$ is isomorphic to the Steinberg representation $\mathbf{St}_G$ of G ([7])

to obtain:

Proposition (4.5). *The G-module $H_c^1(\tilde{X}_0)$ is isomorphic to* $\mathbf{St}_G \oplus \mathrm{c}-\mathrm{ind}_{\tilde{I}}^G \mathbf{1}_{\tilde{I}}$.

5 The inducing representations – II

We now determine the $\mathcal{K}_{s_0}$-module $H^1(\tilde{X}_{2m+1}[s_0])$. The arguments are very often similar to those of the previous section and we will not give all details. Since the case $m = 0$ requires slightly different techniques we postpone it to the end of the section and assume first that $m > 0$.

Recall that the stabilizer $\mathcal{K}_{s_0}$ of s_0 in G is the image K of $\mathrm{GL}(2, \mathfrak{o})$ in G.

Let $c_0 \in C_{2m}(s_0)$ be the path $(s_{-m}, ..., s_0, ..., s_m)$. Its pointwise stabilizer is $\Gamma_0(m,m) = T^0 K_m$. So as a K-module, $H^1(\tilde{X}_{2m+1}[s_0])$ is isomorphic to the

induced representation $\mathrm{Ind}_{T^0K_m}^K H^1(\tilde{X}_{2m+1}[s_0,c_0])$. Moreover the pointwise stabilizer of $\tilde{X}_{2m+1}[s_0,c_0]$ is T^1K_{m+1} and $H^1(\tilde{X}[s_0,c_0])$ may be viewed as a representation of T^0K_m/T^1K_{m+1}.

As in the previous section, one may consider the bipartite graph Ω whose both vertice sets identify with $\mathbf{k}$, equiped with an action of K_m on Ω^1 given by

$$[I_2+\varpi^m\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)].(x,y)=(x+\bar{b},y+\bar{c})\ ,$$

the action of T^0 being given by

$$\left(\begin{array}{cc} a & 0 \\ 0 & d \end{array}\right).(x,y)=(\bar{a}\bar{d}^{-1}x,\bar{d}\bar{a}^{-1}y)\ .$$

Then the contragredient of the T^0K_m/T^1K_{m+1}-module $H^1(\tilde{X}[s_0,c_0])$ is isomorphic to the space $\mathcal{H}(\Omega)$ of harmonic cochains on Ω. As in the previous section this later space is generated by the functions $\chi_1\otimes\chi_2$, where χ_i, $i=1,2$, runs over the non trivial characters of $(k,+)$. The line $\mathbb{C}\chi_1\otimes\chi_2$ is acted upon by K_m via the character $\alpha(\chi_1,\chi_2)$ given by

$$\alpha(\chi_1,\chi_2)\big(I_2+\varpi^m\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\big)=\chi_1(b)\chi_2(a)\ .$$

It follows that $\mathcal{H}(\Omega)$ is isomorphic to its contragredient and that $H^1(\tilde{X}_{2m}[s_0,c_0])$ is the direct sum of the characters $\alpha(\chi_1,\chi_2)$, $\chi_i\not\equiv 1$, $i=1,2$.

For $\chi_i\not\equiv 1$, $i=1,2$, the character $\alpha(\chi_1,\chi_2)$ corresponds to a stratum of the form $[\mathrm{M}(2,\mathfrak{o}),m,m-1,\beta]$, where $\beta\in\mathrm{M}(2,F)$ is given by $\varpi^{-m}\left(\begin{array}{cc} 0 & u \\ v & 0 \end{array}\right)$, u, $v\in\mathfrak{o}^\times$. This stratum is either simple and non-scalar or split fundamental according to whether uv mod $\mathfrak{p}$ is a square in $\mathbf{k}^\times$ or not (here we have used the fact that $\mathrm{Char}\,(\mathbf{k})\neq 2$.

It is clear that T^0 leaves the set of characters corresponding to simple strata (resp. split fundamental strata) stable. So we may write

$$H^1(\tilde{X}_{2m}[s_0,c_0])=H^1(\tilde{X}_{2m}[s_0,c_0])_{\mathrm{simple}}\oplus H^1(\tilde{X}_{2m}[s_0,c_0])_{\mathrm{split}}$$

where $H^1(\tilde{X}_{2m}[s_0,c_0])_{\mathrm{simple}}$ (resp. $H^1(\tilde{X}_{2m}[s_0,c_0])_{\mathrm{split}}$) is the sub-$T^0K_m$-module which decomposes as a K_m/K_{m+1}-module as a direct sum of (characters corresponding to) simple non-scalar strata (resp. split fundamental strata).

We have a result similar to proposition (4.4), whose proof uses the same arguments.

Proposition (5.1). *Let λ be an irreducible constituent of*

$$\mathrm{Ind}_{T^0K_m}^K H^1(\tilde{X}_{2m+1}[s_0,c_0])_{\mathrm{simple}}\subset H^1(\tilde{X}_{2m+1}[s_0])\ .$$

Then the compactly induced representation $c\text{-ind}_K^G\lambda$ *is irreducible, whence supercuspidal.*

The study of $\mathrm{Ind}_{T^0K_m}^K H^1(\tilde{X}_{2m+1}[s_0, c_0])_{\mathrm{split}}$ is the aim of the next section.

We are now going to determine the K-module structure of $H^1(\tilde{X}_1[s_0])$. Set $\mathbf{G} = \mathrm{PGL}(2, \mathbf{k}) \simeq K/K^1$ and write $\mathbf{B}$ and $\mathbf{T}$ for the upper Borel subgroup and diagonal torus of $\mathbf{G}$ respectively. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. As a K-set the set of neighbour vertices of s_0 is isomorpic to $\mathbb{P}^1(\mathbf{k}) = \mathbf{G}/\mathbf{B}$.

The graph $\Omega = \tilde{X}_1[s_0]$ has for vertex set the set of paths of the form (s, s_0) or (s_0, s) where s runs over the neighbour vertices of s_0 in X. So the space $C^0(\Omega)$ of 0-cochains identifies with the space $\mathcal{F}(\mathbb{P}^1(\mathbf{k}) \coprod \mathbb{P}^1(\mathbf{k}))$ of complex valued functions on the disjoint union $\mathbb{P}^1(\mathbf{k}) \coprod \mathbb{P}^1(\mathbf{k})$. So has a $\mathbf{G}$-module $C^0(\Omega)$ is isomorphic to $\mathbf{1_G} \oplus \mathbf{St_G} \oplus \mathbf{1_G} \oplus \mathbf{St_G}$, where $\mathbf{1}$ denotes a trivial representation and $\mathbf{St}$ a Steinberg representation.

The $\mathbf{G}$-set Ω^1 is the set of paths of the form (s, s_0, t), where s and t are two different neighbour vertices of s_0. This $\mathbf{G}$-set is isomorphic to the quotient $\mathbf{G}/\mathbf{T}$. The space $C^{(1)}(\Omega)$ of unoriented 1-cochains identifies as G-module with the space $\mathcal{F}(\mathbf{G}/\mathbf{T})$.

Fix a non-trivial character ψ of $\mathbf{U}$. It is well knows that the induced representation $\mathrm{Ind}_{\mathbf{U}}^{\mathbf{G}}\psi$ is multiplicity free. Its irreducible constituent form by definition the *generic* (irreducible) representations of $\mathbf{G}$. Moreover an irreducible representation is generic if ans only if it is not a character.

We have a natural G-equivariant map $\Phi : \mathcal{F}(\mathbf{G}/\mathbf{T}) \longrightarrow \mathrm{Ind}_{\mathbf{U}}^{\mathbf{G}}\psi$, given by

$$\Phi(f)(g) = \sum_{u \in \mathbf{U}} f(gu)\bar{\psi}(u) \ , \ f \in \mathcal{F}(\mathbf{G}/\mathbf{T}), \ g \in \mathbf{G} \ .$$

If a function f lies in the kernel of Φ, then we have $\sum_{u \in \mathbf{U}} f(gu)\theta(u) = 0$, for all $g \in \mathbf{G}$ an all non-trivial character θ of $\mathbf{U}$. Indeed it suffices to use the fact that the action of $\mathbf{T}$ on $\mathbf{U}$ by conjugation acts transitively on the non-trivial characters of U and the right invariance of f under the action of T. So the kernel of Φ consists of the function f such that $u \mapsto f(gu)$ is constant function on U, for all $g \in G$. In other words $\mathrm{Ker}\,\Phi = \mathcal{F}(G/B) \simeq \mathbf{1_G} \oplus \mathbf{St_G}$. By a dimension argument, we see that Φ is surjective. It follows that

$$C^{(1)}(\Omega) \simeq \mathrm{Ind}_{\mathbf{U}}^{\mathbf{G}}\psi \oplus \mathbf{1_G} \oplus \mathbf{St_G} \ .$$

We have the cochain complex of G-modules:

$$0 \longrightarrow C^0(\Omega) \longrightarrow C^{(1)}(\Omega) \longrightarrow 0$$

Since Ω is connected the kernel of the coboundary operator is the trivial module $\mathbb{C}$. Hence in the Grothendieck groups of G-modules, we have: $dC^0(\Omega) \simeq 2.\mathbf{1_G} + 2.\mathbf{St_G} - \mathbf{1_G} = \mathbf{1_G} + 2.\mathbf{St_G}$. Therefore

$$H^1(\Omega) = C^1(\Omega)/dC^0(\Omega) \simeq \mathrm{Ind}_{\mathbf{U}}^{\mathbf{G}}\psi + \mathbf{1_G} + \mathbf{St_G} - \mathbf{1_G} - 2.\mathbf{St_G} = \mathrm{Ind}_{\mathbf{U}}^{\mathbf{G}}\psi - \mathbf{St_G} \ .$$

Since $q = |\mathbf{k}|$ is odd, there exists a unique non-trivial character of $\mathbf{k}^\times/(\mathbf{k}^\times)^2$, that we denote by χ_0. The irreducible constituents of the Gelfand-Graev representation $\mathrm{Ind}_{\mathbf{U}}^{\mathbf{G}}\psi$ are the following:

– the irreducible cuspidal representations of $\mathbf{G}$,

– the principal series $\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}\chi\otimes\chi^{-1}$, where $\chi : \mathbf{k}^\times \longrightarrow \mathbb{C}^\times$ is a character such that $\chi^2 \not\equiv 1$ (i.e. $\chi \notin \{\mathbf{1}, \chi_0\}$).

– the steinberg representation $\mathbf{St_G}$,

– (when q is odd) the twisted representation $\mathbf{St_G}\otimes\chi_0$.

If σ is a cuspidal representation of $\mathbf{G} = K/K^1$, then the induced representation $c\text{-ind}_K^G\sigma$ is irreducible and supercuspidal ([4], (11.5), page 81). Such a representation of G is called a *level* 0 *supercuspidal* representation.

A principal series of $\mathbf{G} = K/K^1$ may be writen as $\mathrm{Ind}_I^K\rho$, where ρ is a character of I/I^1. The pair (I,ρ) is actually a type in the sense of Bushnell and Kutzko's type theory. For technical reason we postpone definitions and references to the next section. Since the representation $\mathrm{Ind}_I^K\rho$ is irreducible, it is a type for the same constituent as (I,ρ).

To sum up, we have proved the following.

Proposition (5.2). *An irreducible constituent λ of $H^1(\tilde{X}_1[s_0])$ is of one of the following forms*

(i) *the inflation of a cuspidal representation of* $\mathbf{G}$*; in that case* $c\text{-ind}_K^G\lambda$ *is a level* 0 *irreducible supercuspidal representation of G.*

(ii) *the inflation to K of the representation* $\mathbf{St_G}\otimes\chi_0$*,*

(iii) *a type of the form $\mathrm{Ind}_I^K\rho$, where the ρ is inflated from a character of $I/I^1 \simeq (\mathbf{k}^\times\times\mathbf{k}^\times)/\mathbf{k}^\times$ of the form $\chi\otimes\chi^{-1}$, $\chi^2\not\equiv 1$.*

Note that in (iii), the pair $(K, \mathrm{Ind}_I^K\,\rho)$ is a principal series type.

6 The inducing representations – III

We keep the notation as in the previous section. To determine the structure of $\mathrm{Ind}_{T^0K_m}^K H^1(\tilde{X}_{2m+1}[s_0,c_0])_{\mathrm{split}}$, we first recall crucial facts on split strata and types for principal series representations. The basic reference for *type theory* is [5].

Let χ be a character of T, that we view as a character of T^0 by restriction. Assume that the conductor of χ is $n > 0$: $T^n \subset \operatorname{Ker}\chi$ and n is minimal for this property. Set

$$J_\chi = \begin{pmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o}^\times \end{pmatrix} = \Gamma_0(\mathfrak{p}^n) \ .$$

If U and $\bar{U}$ denotes the groups of upper and lower unipotent matrices respectively, then J_χ has an Iwahori decomposition:

$$J_\chi = (J_\chi \cap \bar{U}).(J_\chi \cap T).(J_\chi \cap U)$$

and one may define a character ρ_χ of J_χ by

$$\rho_\chi(\bar{u}t^0 u) = \chi(t^0) \ , \ \bar{u} \in J_\chi \cap \bar{U}, \ u \in J_\chi \cap U, \ t^0 \in T^0 \ .$$

Let $\mathcal{R}_{[T,\chi]}$ be the Bernstein component of the category of smooth representations of G whose objects are the representations $\mathcal{V}$ satisfying the following property : any irreducible subquotient of $\mathcal{V}$ occurs in a parabolically induced representation $\operatorname{Ind}_{\mathcal{B}}^G(\chi \otimes \chi_0)$, where $\mathcal{B}$ is a Borel subgroup with Levi component T and χ^0 an unramified character of T. We then have.

Theorem (6.1) (A. Roche) *The pair (J_χ, ρ_χ) is a type for $\mathcal{R}_{[T,\chi]}$.*

This is indeed Theorem (7.7) of [11]. Note that our J_χ is not exactly Roche's one, but a conjugate under an element of T (see [11], Example (3.5)).

Proposition (6.2). *With the notation as before, assume that $\chi_{|T^0}$ is not of the form $\alpha \circ \operatorname{Det}$, where α is a character of $\mathfrak{o}^\times$ (necessarily of order 2). Then the induced representation $\operatorname{Ind}_{J_\chi}^K \rho_\chi$ is irreducible. In particular it is a type for $\mathcal{R}_{[T,\chi]}$.*

Proof. Let W be the extended affine Weyl group of G w.r.t. T and set $W_\chi = \{w \in W \ ; \ w\chi = \chi\}$. Then by Theorem (4.14) of [11], the G-intertwining of ρ_χ is $J_\chi W_\chi J_\chi$. The hypothesis on χ forces $W_\chi = T/T^0$. So $(J_\chi W_\chi J_\chi) \cap K = J_\chi T^0 J_\chi = J_\chi$, and we may apply Mackey's criterion of irreducibility.

For $n > 0$ and $q \in \{0, ..., n\}$, define compact open subgroups of G as follows:

$$_q\mathfrak{h}_1 = \begin{pmatrix} 1+\mathfrak{p}^n & \mathfrak{p}^q \\ \mathfrak{p}^{n+1} & 1+\mathfrak{p}^n \end{pmatrix} \text{ and } {_q\mathfrak{h}_2} = \begin{pmatrix} 1+\mathfrak{p}^{n+1} & \mathfrak{p}^q \\ \mathfrak{p}^{n+1} & 1+\mathfrak{p}^{n+1} \end{pmatrix} \ .$$

These groups are particular cases of groups considered in [1], §(2.3). The quotients ${_q\mathfrak{h}_1}/{_q\mathfrak{h}_2}$, $q = 0, ..., n$, are abelian, and for $\alpha \in \mathbf{k}^\times$, one may define a character ψ_α of ${_q\mathfrak{h}_1}/{_q\mathfrak{h}_2}$ by the formula:

$$\psi_\alpha(I_2 + \begin{pmatrix} \varpi^n a & \varpi^q b \\ \varpi^{n+1} c & \varpi^n d \end{pmatrix}) = \psi(\alpha(a-d))$$

where ψ is a fixed non-trivial character of $(\mathbf{k},+)$. In fact, $(\psi_\alpha)_{|_n\mathfrak{h}_1}$ is the restriction to ${}_n\mathfrak{h}_1$ of a split fundamental stratum of K_n/K_{n+1}. We shall need the following result.

Lemma (6.3). *If a smooth representation of K contains $(\psi_\alpha)_{|_n\mathfrak{h}_1}$ by restriction, then it contains the character $(\psi_\alpha)_{|_0\mathfrak{h}_1}$.*

Proof. Since the characteristic of $\mathbf{k}$ is not 2, then $\alpha \neq -\alpha$ $(\psi_\alpha)_{|_n\mathfrak{h}_1}$ is the restriction to ${}_n\mathfrak{h}_1$ of a split fundamental stratum of K_n/K_{n+1}. Our lemma is then a particular case of [1], Lemma (2.4.5).

Proposition (6.4). *Let λ be an irreducible constituent of* $\mathrm{Ind}_{T^0K_m}^{K} H^1(\tilde{X}_{2m+1}[s_0,c_0])_{split}$. *Then with the notation as above, λ is of the form $\mathrm{Ind}_{J_\chi}^{K}\rho_\chi$, for some principal series type (J_χ,ρ_χ) with χ of conductor $m+1$.*

Proof. We know that such a λ contains a split fundamental stratum of the form $[\mathrm{M}(2,\mathfrak{o}),m,m-1,b]$, where $b=\varpi^{-m}\begin{pmatrix} 0 & u \\ v & 0\end{pmatrix}$, $u,v\in\mathfrak{o}^\times$, and uv is a square *modulo* $\mathfrak{p}$. If $\alpha\in\mathfrak{o}$ is such that $\alpha^2\equiv uv\ mod\ \mathfrak{p}$, then the stratum is equivalent to a K-conjugate of $[\mathrm{M}(2,\mathfrak{o}),m,m-1,b']$, where $b'=\varpi^{-m}\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha\end{pmatrix}$. So we deduce that λ contains this latter stratum by restriction. Now consider the group ${}_q\mathfrak{h}_1$ for $n=m$. The representation λ contains the character $(\psi_\alpha)_{|_n\mathfrak{h}_1}$ by restriction. By applying Lemma (6.3) we obtain that it contains the character $(\psi_\alpha)_{|_0\mathfrak{h}_1}$. This character clearly extends to $T^0{}_0\mathfrak{h}_1=\Gamma_0(m+1,0)$ and the quotient $T^0{}_0\mathfrak{h}_1/{}_0\mathfrak{h}_1$ is abelian. It follows that λ contains and extension of ψ_α to $\Gamma_0(m+1,0)$. Such an extension is of the form (J_χ,ρ_χ), for some character χ of T of conductor $m+1$. The fact that λ is induced from (J_χ,ρ_χ) follows from Proposition (6.2).

7 Synthesis

We now prove Theorems A and B of the introduction.

By Proposition (3.5) and (3.11), we have isomorphisms of G modules :

$$H^1_c(\tilde{X}_{2m})\simeq H^1_c(\tilde{X}_{2m-1})\oplus c\text{-}\mathrm{ind}_{\mathcal{K}_1}^{G} H^1(\Sigma_{2m}),\ m\geqslant 1. \tag{1}$$

$$H^1_c(\tilde{X}_{2m=1})\simeq H^1_c(\tilde{X}_{2m})\oplus c\text{-}\mathrm{ind}_{\mathcal{K}_0}^{G} H^1(\Sigma_{2m+1}),\ m\geqslant 0. \tag{2}$$

Recall that with the notation of the introduction, we have :

– $\Sigma_{2m}=(\tilde{X}_{2m})_{e_0}$, $\Sigma_{2m+1}=(\tilde{X}_{2m+1})_{s_0}$,
– $\mathcal{K}_0=\mathcal{K}_{s_0}$, $\mathcal{K}_1=\mathcal{K}_{e_0}$.

Moreover, by Proposition (4.5), we have

$$H^1_c(\tilde{X}_0) \simeq \mathrm{St}_G \oplus c\text{-}\mathrm{ind}_{\mathcal{K}_1}^G H^1(\Sigma_0) \tag{3}$$

so that (1) holds for $m = 0$. Hence Theorem A follows from (1) and (2) by a straightforward inductive argument.

Theorem B follows from the discription of the irreducible components of $H^1(\Sigma_n)$ given in Proposition (4.4) (n even and $n > 0$), Proposition (4.5) ($n = 0$), and Propositions (5.1) and (6.4) (n odd).

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Laboratoire de Mathématiques et
UMR 7348 CNRS

SP2MI - Téléport 2
Bd M. et P. Curie BP 30179
86962 Futuroscope Chasseneuil Cedex
France

E-mail : paul.broussous@math.univ-poitiers.fr